Transformations and Consistency Relations of Distillation Control Structures

Rigorous analytical methods for consistent modeling of control structures for multivariable processes are presented. The continuous distillation process is used to illustrate the methods, but the basic ideas, and more specifically the transformations and consistency relations derived, are valid in general.

If steady-state operating data and the process gains of an arbitrary control structure are known, it is possible to calculate the process gains of any feasible control structure. A general expression relating the process gains of different control structures is derived. In general, the process gains must also satisfy certain consistency relationships which can be derived from first principles, e.g., steady-state material balances.

The usefulness of the results is illustrated by control structure transformations and reconciliation of process gains, by an application to process dynamics, by synthesis of noninteracting control loops, and by derivation of analytical relationships useful in relative gain analysis.

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Introduction

One of the crucial problems in process control is the synthesis of control system structures. There is a great demand for systematic methods of studying various structures by means of the properties of one known structure.

In chemical process control, true multivariable control methods like linear quadratic control have not been generally accepted in industry. The usual approach to control multivariable processes in the process industries is to use multiloop SISO control. The synthesis of control structures then consists of two parts. The first is the choice of manipulators and controlled variables to be used in the system, the second is how to pair the inputs with the outputs.

Instead of controlling "pure" outputs by "pure" flows, a number of "mixed" inputs and outputs have been suggested. In distillation, the use of various flow ratios as manipulators has been recommended for a long time [an early summary is given in Rademaker et al. (1975, p. 445)]. Also control of the sums and differences of compositions or temperatures instead of the compositions or temperatures themselves has been suggested [recent reviews are given in Waller (1986) and Waller and Finnerman (1987)].

Using various functions of inputs and outputs as manipulators and controlled variables generally means that the resulting control system is a multivariable system although it has been designed as a multiloop SISO system (Waller, 1986).

The main criterion behind control synthesis according to the methods outlined above has usually been to minimize interaction between the SISO loops. Another important difference is, however, that the various structures may have a very different sensitivity to disturbances: to various degrees the different structures are self-regulating (Shinskey, 1985; Skogestad and Morari, 1987b; Waller et al., 1986, 1988). The self-regulating properties are also different for different disturbances.

In the investigation of control schemes for process control it has usually not been recognized that the process gains, and even complete transfer functions, of different control schemes for the same process are analytically related to each other, nor that the process gains of individual control schemes must satisfy certain consistency relationships. The relationships between different control schemes exist for purely mathematical reasons (variable and system transformations), while the consistency relationships of individual control schemes have their origin in physical principles (conservation laws).

In this paper it is shown how the process gains of different control schemes are related to one another. A general expression for transformations between control structures is derived. Consistency relationships existing between the process gains of individual control structures are also derived.

The paper is mainly focused on the steady-state part, i.e. the gains, of the transfer functions describing the continuous distillation process. The steady-state part contains much information about the process as recently shown by Grosdidier et al. (1985) and previously by, e.g., Bristol (1966), Wahl and Harriott (1970) and Waller (1972, 1979).

The control structure transformations and consistency relationships are useful for many purposes. They can be utilized, e.g., to

- Investigate a large number of control systems once the properties of one system are known
- Reduce experimental and simulation work
- Improve accuracy in process gain estimation
- Remove doubtful assumptions which reduce the reliability of the results

The applications of control structure transformations and consistency relationships treated in this paper are process gain calculation and reconciliation, process dynamics, synthesis of noninteracting control systems (one-way and two-way internal decoupling), and relative gain analysis.

For simplicity, the relations are derived for a binary (or one that can be treated as binary) two-product distillation process. It is, however, fairly straightforward to generalize the treatment to include more complex distillation processes with an arbitrary number of components, feeds, and sidestreams (Häggblom, 1986).

Another simplification made in the derivation of control structure transformations and consistency relationships in this paper is that disturbances are not included in the models used to describe the control structures. However, the general expressions, derived using matrix notation, apply also when disturbances are considered. In Häggblom (1987), Häggblom and Waller (1988), and Waller et al. (1988) the disturbances are explicitly taken into account in control structure transformations and consistency relationships.

Consistency and Transformations: Introductory Examples

Consistency relationships

Consider the distillation process in Figure 1. Continuous operation requires control of the column inventory. This is usually accomplished by controlling the holdup in the reboiler and the reflux drum. Hence, the material balances

$$F = D + B \tag{1a}$$

$$Fz = Dy + Bx \tag{1b}$$

must hold at steady state. Linearization of Eqs. 1 when F and z are constant (only relations between the product compositions and the product flows are considered here) gives

$$\Delta D + \Delta B = 0 \tag{2a}$$

$$\bar{y}\Delta D + \bar{D}\Delta y + \bar{x}\Delta B + \bar{B}\Delta x = 0 \tag{2b}$$

or

$$\Delta D = -\Delta B = -\frac{\bar{D}}{\bar{y}-\bar{x}}\Delta y - \frac{\bar{B}}{\bar{y}-\bar{x}}\Delta x \tag{2c}$$

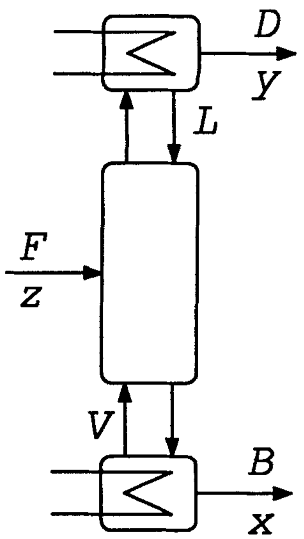

Figure 1. Two-product distillation process.

where Δ denotes a (small) steady-state deviation from the nominal steady state defined by $\bar{D}$, $\bar{B}$, $\bar{y}$, and $\bar{x}$.

Next, consider the "energy balance" or "conventional" control scheme where the top composition y and the bottoms composition x are controlled by the reflux flow L and the boilup V, or by any two independent variables directly affecting the reflux flow and the boilup (V could, e.g., stand for heat input to the reboiler in the equations that follow). A steady-state model for this structure is

$$\Delta y = K_{yL}^{LV}\Delta L + K_{yV}^{LV}\Delta V \tag{3a}$$

$$\Delta x = K_{xL}^{LV}\Delta L + K_{xV}^{LV}\Delta V \tag{3b}$$

Elimination of Δy and Δx in Eq. 2c by Eqs. 3 gives

$$\Delta D = K_{DL}^{LV}\Delta L + K_{DV}^{LV}\Delta V \tag{4a}$$

$$\Delta B = K_{BL}^{LV}\Delta L + K_{BV}^{LV}\Delta V \tag{4b}$$

where

$$K_{DL}^{LV} = -K_{BL}^{LV} = -\frac{\bar{D}K_{yL}^{LV} + \bar{B}K_{xL}^{LV}}{\bar{y}-\bar{x}} \tag{5a}$$

$$K_{DV}^{LV} = -K_{BV}^{LV} = -\frac{\bar{D}K_{yV}^{LV} + \bar{B}K_{xV}^{LV}}{\bar{y}-\bar{x}} \tag{5b}$$

Equations 4 and 5 are fundamental steady-state relations between the composition manipulators L, V and the inventory manipulators D, B in the conventional control scheme (L, V).

The distillation model for the (L, V) structure consists of both Eqs. 3 and 4, where the gains are related by the consistency relationships expressed by Eqs. 5. For a complete definition of the steady-state model when product compositions are the *primary* output variables it is thus sufficient to give the four gains in Eqs. 3 together with steady-state data.

Where models for the conventional scheme are given in the literature, the four gains in Eqs. 3 are reported, but steady-state data are often missing. Even if the steady-state data are available, the usual approach in the literature is to use

$$K_{DL}^{LV} = -K_{BL}^{LV} = -1 \text{ (or } \Delta D = -\Delta L,\ V = \text{const.)} \quad (6a)$$

$$K_{DV}^{LV} = -K_{BV}^{LV} = 1 \text{ (or } \Delta D = \Delta V,\ L = \text{const.)} \quad (6b)$$

where V in Eq. 6b specifically denotes vapor boilup. These relations are, however, valid under very special circumstances only and almost never in real columns. Numerical illustrations are given later in this paper.

When the four gains in Eqs. 3 are estimated, e.g., from experimental or simulated data, they should be determined in such a way that also Eqs. 4, with the gains calculated by Eqs. 5, agree well with the data in question. More precisely, the task is to determine the eight gains in Eqs. 3 and 4 subject to the constraints imposed by Eqs. 5.

If the primary output variables are not product compositions, but, e.g., two temperatures in the column, the gains in Eqs. 4 cannot be calculated by Eqs. 5, because the gains of the model replacing Eqs. 3 are no longer related to composition. In this case the gains in Eqs. 4 have to be determined independently of the four gains between the manipulators and the primary output variables. Then the variable transformations presented below can be used even if the output variables are not product compositions. If only the four gains between the manipulators and the output variables are given, the model is simply not complete enough in this case for arbitrary variable transformations.

The example above can be generalized. For every control structure where two independent functions of L and V (and possibly D or B) are used as composition manipulators, consistency relationships analogous with Eqs. 5 exist. Such control structures are, e.g., Ryskamp's control structure $[D/(L + D), V]$ (Ryskamp, 1980), treated in Häggblom and Waller (1986), and the two-ratio structure $[D/(L + D), V/B]$ suggested by Shinskey (1984) and Takamatsu et al. (1982, 1984).

Next, consider a control structure where the product compositions are controlled by manipulation of D and V, a so called material balance scheme (Shinskey, 1984). A model for this structure is given by the following equations:

$$\Delta y = K_{yD}^{DV} \Delta D + K_{yV}^{DV} \Delta V \quad (7a)$$

$$\Delta x = K_{xD}^{DV} \Delta D + K_{xV}^{DV} \Delta V \quad (7b)$$

$$\Delta L = K_{LD}^{DV} \Delta D + K_{LV}^{DV} \Delta V \quad (8a)$$

$$\Delta B = K_{BD}^{DV} \Delta D + K_{BV}^{DV} \Delta V \quad (8b)$$

There is an important principal difference between Eqs. 3 and Eqs. 7: Eqs. 7 contain redundant information while Eqs. 3 do not. This can be shown in the following way.

Combination of Eqs. 7 with Eq. 2c gives

$$(\bar{D} K_{yD}^{DV} + \bar{B} K_{xD}^{DV} + \bar{y} - \bar{x}) \Delta D + (\bar{D} K_{yV}^{DV} + \bar{B} K_{xV}^{DV}) \Delta V = 0 \quad (9)$$

As D and V are independent variables in the material balance control scheme, Eq. 9 must be valid regardless of the values of ΔD and ΔV. This means that

$$\bar{D} K_{yD}^{DV} + \bar{B} K_{xD}^{DV} + y - \bar{x} = 0$$
$$\bar{D} K_{yV}^{DV} + \bar{B} K_{xV}^{DV} = 0$$

or

$$K_{xD}^{DV} = -\frac{\bar{D}}{\bar{B}} \left(K_{yD}^{DV} + \frac{\bar{y} - \bar{x}}{\bar{D}} \right) \quad (10a)$$

$$K_{xV}^{DV} = -\frac{\bar{D}}{\bar{B}} K_{yV}^{DV} \quad (10b)$$

must hold.

Equations 10 express consistency relationships between the gains in the model for the material balance scheme (D, V). Thus, there are only two independent process gains in Eqs. 7. If the gains in Eqs. 7 are determined from steady-state data, they should therefore be determined subject to the constraints imposed by Eqs. 10.

Equations 7 do not give any information about the effect of L on the product compositions, not even when the material balances are taken into account. This means that the model as given by Eqs. 7 cannot be transformed to a model containing L as a composition manipulator. To overcome this, the relation between L and the manipulators D and V, i.e., Eq. 8a, should be included in the model for the material balance scheme. Then a model for, e.g., the (L, V) structure can be derived by elimination of ΔD from Eqs. 7 by Eq. 8a.

However, when process models for the (D, V) structure are given in the literature, the process gains for the flow dynamics, Eqs. 8, are usually omitted. This is unfortunate, as the process gains in Eqs. 8 cannot be derived from the process gains in Eqs. 7, and the model cannot thus be used for arbitrary control structure transformations.

For any control structure where product compositions are used as primary output variables it is, in fact, sufficient to give one of the composition relationships and one of the flow relationships, together with steady-state data, in order to define the steady-state model completely.

Control structure transformations

Not only are there consistency relationships between gains in single control schemes, there are also fundamental relations between gains in different control schemes.

Consider the model for the conventional control scheme (L, V), Eqs. 3. Elimination of ΔL by Eq. 4a gives

$$\Delta y = \frac{K_{yL}^{LV}}{K_{DL}^{LV}} \Delta D + \left(K_{yV}^{LV} - \frac{K_{DV}^{LV}}{K_{DL}^{LV}} K_{yL}^{LV} \right) \Delta V \quad (11a)$$

$$\Delta x = \frac{K_{xL}^{LV}}{K_{DL}^{LV}} \Delta D + \left(K_{xV}^{LV} - \frac{K_{DV}^{LV}}{K_{DL}^{LV}} K_{xL}^{LV} \right) \Delta V \quad (11b)$$

which now is a model for the material balance scheme (D, V). If one compares Eqs. 11 and 7 and takes Eqs. 5 into account, the following relations between the gains in the material balance scheme and the conventional scheme are obtained:

$$K_{yD}^{DV} = \frac{K_{yL}^{LV}}{K_{DL}^{LV}} = -\frac{(\bar{y} - \bar{x}) K_{yL}^{LV}}{\bar{D} K_{yL}^{LV} + \bar{B} K_{xL}^{LV}} \quad (12a)$$

$$K_{yV}^{DV} = K_{yV}^{LV} - \frac{K_{DV}^{LV}}{K_{DL}^{LV}} K_{yL}^{LV} = -\frac{\bar{B}(K_{yL}^{LV} K_{xV}^{LV} - K_{xL}^{LV} K_{yV}^{LV})}{\bar{D} K_{yL}^{LV} + \bar{B} K_{xL}^{LV}} \quad (12b)$$

$$K_{xD}^{DV} = \frac{K_{xL}^{LV}}{K_{DL}^{LV}} = -\frac{(\bar{y} - \bar{x})K_{xL}^{LV}}{\bar{D}K_{yL}^{LV} + \bar{B}K_{xL}^{LV}} \quad (12c)$$

$$K_{xV}^{DV} = K_{xV}^{LV} - \frac{K_{DV}^{LV}}{K_{DL}^{LV}}K_{xL}^{LV} = \frac{\bar{D}(K_{yL}^{LV}K_{xV}^{LV} - K_{xL}^{LV}K_{yV}^{LV})}{\bar{D}K_{yL}^{LV} + \bar{B}K_{xL}^{LV}} \quad (12d)$$

Naturally, Eqs. 12 satisfy the consistency relations (Eqs. 10).

In a similar manner the gains of any feasible control scheme can be derived from the gains of any control scheme adequately modeled. It is, however, not uncommon that reported gains do not satisfy fundamental consistency relationships like Eqs. 12.

Transformations of Distillation Control Structures

In the previous section it was illustrated how the process gains of the control structure (D, V) can be calculated from known gains of the control structure (L, V). In this section a more general treatment is given. The subject of control system transformations has recently been treated also by Skogestad and Morari (1987a).

General transformation

Define

$$\boldsymbol{y} = [y \quad x]^T \quad (13)$$

$$\boldsymbol{u} = [L \quad V]^T \quad (14)$$

$$\boldsymbol{v} = [D \quad B]^T \quad (15)$$

$$K_{yu} = \begin{bmatrix} K_{yL}^{LV} & K_{yV}^{LV} \\ K_{xL}^{LV} & K_{xV}^{LV} \end{bmatrix} \quad (16)$$

$$K_{vu} = \begin{bmatrix} K_{DL}^{LV} & K_{DV}^{LV} \\ K_{BL}^{LV} & K_{BV}^{LV} \end{bmatrix} \quad (17)$$

The model for the (L, V) structure, expressed by Eqs. 3 and 4, can then be compactly written

$$\begin{bmatrix} \Delta \boldsymbol{y} \\ \Delta \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} K_{yu} \\ K_{vu} \end{bmatrix} \Delta \boldsymbol{u} \quad (18)$$

Assume that one wants to use $\boldsymbol{\mu}$ instead of $\boldsymbol{u}$ as manipulator to control $\boldsymbol{y}$ and that the manipulators used for level control are $\boldsymbol{\nu}$ instead of $\boldsymbol{v}$. The linear model having these manipulators can then be written

$$\begin{bmatrix} \Delta \boldsymbol{y} \\ \Delta \boldsymbol{\nu} \end{bmatrix} = \begin{bmatrix} K_{y\mu} \\ K_{\nu\mu} \end{bmatrix} \Delta \boldsymbol{\mu} \quad (19)$$

where $\boldsymbol{\mu} = [\mu_1 \quad \mu_2]^T$, $\boldsymbol{\nu} = [\nu_1 \quad \nu_2]^T$, and $K_{y\mu}$ and $K_{\nu\mu}$ are gain matrices defined analogously with Eqs. 16 and 17. As $\boldsymbol{u} = [L \quad V]^T$ and $\boldsymbol{v} = [D \quad B]^T$ still are the only physical manipulators available, $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ must be functions of these, i.e.,

$$\boldsymbol{\mu} = \boldsymbol{\mu}(\boldsymbol{u}, \boldsymbol{v}) \quad (20a)$$

$$\boldsymbol{\nu} = \boldsymbol{\nu}(\boldsymbol{u}, \boldsymbol{v}) \quad (20b)$$

Since we are here considering only linear models, Eqs. 20 have to be linearized (if they are nonlinear). The linearization and introduction of deviation variables result in the following relation

$$\begin{bmatrix} \Delta \boldsymbol{\mu} \\ \Delta \boldsymbol{\nu} \end{bmatrix} = \begin{bmatrix} H_{\mu u} & H_{\mu v} \\ H_{\nu u} & H_{\nu v} \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{u} \\ \Delta \boldsymbol{v} \end{bmatrix} \quad (21)$$

where $H_{\mu u}$, $H_{\mu v}$, $H_{\nu u}$, and $H_{\nu v}$ are matrices of partial derivatives that define the variable transformation in linear terms.

Elimination of $\Delta \boldsymbol{v}$ from Eq. 21 by Eq. 18 gives

$$\begin{bmatrix} \Delta \boldsymbol{\mu} \\ \Delta \boldsymbol{\nu} \end{bmatrix} = \begin{bmatrix} H_{\mu u} + H_{\mu v}K_{vu} \\ H_{\nu u} + H_{\nu v}K_{vu} \end{bmatrix} \Delta \boldsymbol{u} \quad (22)$$

Solving Eq. 22 for the relation between $\Delta \boldsymbol{u}$ and $\Delta \boldsymbol{\mu}$ gives

$$\Delta \boldsymbol{u} = [H_{\mu u} + H_{\mu v}K_{vu}]^{-1} \Delta \boldsymbol{\mu} \quad (23)$$

For any feasible control structure, where $\dim(\boldsymbol{\mu}) = \dim(\boldsymbol{u})$, the inverse in Eq. 23 must exist. Otherwise $\boldsymbol{u}$ and $\boldsymbol{\mu}$ would not be vectors of independent variables and would not be usable as manipulators. A more practical explanation is that $\boldsymbol{u}$ must be determinable from $\boldsymbol{\mu}$ because $\boldsymbol{u}$ contains the variables by which the process is physically manipulated.

Inserting Eq. 23 into Eqs. 18 and 22 gives the desired model, Eq. 19, with

$$\begin{bmatrix} K_{y\mu} \\ K_{\nu\mu} \end{bmatrix} = \begin{bmatrix} K_{yu} \\ H_{\nu u} + H_{\nu v}K_{vu} \end{bmatrix} [H_{\mu u} + H_{\mu v}K_{vu}]^{-1} \quad (24)$$

The expression for the control system transformation, Eq. 24, has above been derived starting from the conventional (L, V) structure. This was done in order to make the treatment more perspicious. However, no actual use of the definitions of $\boldsymbol{u}$ and $\boldsymbol{v}$ (Eqs. 14 and 15) was made. Also, the output vector $\boldsymbol{y}$ could denote, e.g., temperatures instead of compositions.

Furthermore, the only assumption regarding the dimension of the input and output vectors is that $\dim(\boldsymbol{\mu}) = \dim(\boldsymbol{u})$. This means that the transformation is valid in general for a separation with an arbitrary number of components, feeds, and sidestreams when the input and output vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{y}$ in the model, Eq. 18, are appropriately defined (Häggblom, 1986). The transformation is, in fact, valid for any process that can be described by models in the form of Eq. 18.

In the derivation above only transformations of input variables only were considered. It is, however, straightforward to include also transformations of output variables, as shown in Häggblom (1986).

Strictly, the system transformation (Eq. 24) applies to process gains only, i.e., the steady-state parts of the transfer functions. However, if perfect inventory control in a dynamic sense can be assumed in the control systems considered, Eq. 24 is valid also for the dynamic case when the process gains are replaced by transfer functions (Häggblom, 1986). (If the transformation results in an unrealizable control structure, e.g., because of a badly placed dead time, it means that perfect inventory control cannot be realized even in principle by feedback control.)

The assumption of perfect inventory control is often a good approximation in distillation, where the level control loops usually are much faster than the composition (or temperature) control loops.

Next is illustrated, by two examples, the use of the transformation (Eq. 24) in deriving the process gains of a control structure in terms of the process gains of another control structure.

(D, V) structure

A model for the material balance control scheme with $\boldsymbol{\mu} = [D\ V]^T$ and $\boldsymbol{\nu} = [L\ B]^T$ is obtained from a model for the conventional scheme with $\boldsymbol{u} = [L\ V]^T$ and $\boldsymbol{v} = [D\ B]^T$ in the following way.

Defining the manipulators of the (D, V) scheme in terms of the manipulators of the (L, V) scheme in accordance with Eq. 21, gives the relations

$$\begin{bmatrix}\Delta D\\ \Delta V\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\Delta L\\ \Delta V\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\Delta D\\ \Delta B\end{bmatrix} \tag{25a}$$

$$\begin{bmatrix}\Delta L\\ \Delta B\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\Delta L\\ \Delta V\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\Delta D\\ \Delta B\end{bmatrix} \tag{25b}$$

which show the definitions of the transformation matrices $\boldsymbol{H}$ in Eq. 21.

The process gains of the (D, V) structure are then obtained by means of Eq. 24 and the definitions in Eqs. 16 and 17. The result is

$$\boldsymbol{K}_{y\mu} = \begin{bmatrix}K_{yD}^{DV} & K_{yV}^{DV}\\ K_{xD}^{DV} & K_{xV}^{DV}\end{bmatrix} = \frac{1}{K_{DL}^{LV}}\begin{bmatrix}K_{yL}^{LV} & -(K_{yL}^{LV}K_{DV}^{LV} - K_{yV}^{LV}K_{DL}^{LV})\\ K_{xL}^{LV} & -(K_{xL}^{LV}K_{DV}^{LV} - K_{xV}^{LV}K_{DL}^{LV})\end{bmatrix} \tag{26a}$$

$$\boldsymbol{K}_{\nu\mu} = \begin{bmatrix}K_{LD}^{DV} & K_{LV}^{DV}\\ K_{BD}^{DV} & K_{BV}^{DV}\end{bmatrix} = \frac{1}{K_{DL}^{LV}}\begin{bmatrix}1 & -K_{DV}^{LV}\\ K_{BL}^{LV} & -(K_{BL}^{LV}K_{DV}^{LV} - K_{BV}^{LV}K_{DL}^{LV})\end{bmatrix} \tag{26b}$$

Equation 26a gives the same relationships between the process gains of the two control structures as the first parts of the previously derived Eqs. 12. If the output variables are product compositions, the consistency relations expressed by Eqs. 5 will hold. Taking these into account will, of course, result in the second parts of Eqs. 12.

According to the first parts of Eqs. 5, K_{BL}^{LV} and K_{BV}^{LV} can be replaced by $-K_{DL}^{LV}$ and $-K_{DV}^{LV}$, respectively, regardless of which the output variables are. From this follows that $K_{BV}^{DV} = -(K_{BL}^{LV}K_{DV}^{LV} - K_{BV}^{LV}K_{DL}^{LV})/K_{DL}^{LV} = 0$ in Eq. 26b, as it, of course, should be.

The matrix to be inverted in Eq. 24 for the (D, V) scheme is

$$\boldsymbol{H}_{\mu u} + \boldsymbol{H}_{\mu v}\boldsymbol{K}_{vu} = \begin{bmatrix}K_{DL}^{LV} & K_{DV}^{LV}\\ 0 & 1\end{bmatrix} \tag{27}$$

Since it is hard to imagine a distillation process with $K_{DL}^{LV} = 0$, i.e., a process where a change in L would not ultimately result in a change in D when V is kept constant, the inverse can safely be assumed to exist.

[D/(L + D), V] structure

Consider the $[D/(L + D), V]$ scheme suggested by Ryskamp (1980). The scheme has usually been studied with $\nu_1 = L + D$, but, e.g., L or D can also be chosen for ν_1. In the sequel $\nu_1 = L + D$ is used. Then the manipulators are

$$\boldsymbol{\mu} = \begin{bmatrix}D/(L + D)\\ V\end{bmatrix} \quad \boldsymbol{\nu} = \begin{bmatrix}L + D\\ B\end{bmatrix} \tag{28}$$

The variable transformation can be written in linearized form as

$$\begin{bmatrix}\Delta(D/(L + D))\\ \Delta V\end{bmatrix} = \begin{bmatrix}-\bar{D}/(\bar{L} + \bar{D})^2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\Delta L\\ \Delta V\end{bmatrix} + \begin{bmatrix}\bar{L}/(\bar{L} + \bar{D})^2 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\Delta D\\ \Delta B\end{bmatrix} \tag{29a}$$

$$\begin{bmatrix}\Delta(L + D)\\ \Delta B\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\Delta L\\ \Delta V\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\Delta D\\ \Delta B\end{bmatrix} \tag{29b}$$

where $\bar{L}$ and $\bar{D}$ denote the nominal steady-state at which the linearization is made.

Equations 29 thus give the transformation matrices $\boldsymbol{H}$ in Eq. 21. The gains in the transfer functions for the $[D/(L + D), V]$ structure with the manipulators as defined by Eq. 28 are then obtained by means of Eq. 24.

Another choice for $\boldsymbol{\nu}$ would result in a different gain matrix $\boldsymbol{K}_{\nu\mu}$, while $\boldsymbol{K}_{y\mu}$, which usually is the more interesting one, would remain unchanged.

Infeasible control structures

The transformation expressed by Eq. 24 presupposes that the inverse in Eq. 24 exists. As shown above, the inverse exists for all transformations into structures where the free manipulators can be independently adjusted without violating the material balances at steady state, but not for transformations into structures where this is not so. The latter are clearly infeasible control structures which, if implemented, would result in a nonoperable column.

Suppose that one would try to choose D and B as manipulators to control composition. Then one would have $\boldsymbol{\mu} = [D\ B]^T$. With $\boldsymbol{u} = [L\ V]^T$ and $\boldsymbol{v} = [D\ B]^T$ one would get

$$\boldsymbol{H}_{\mu u} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \boldsymbol{H}_{\mu v} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

and

$$\boldsymbol{H}_{\mu u} + \boldsymbol{H}_{\mu v}\boldsymbol{K}_{vu} = \boldsymbol{K}_{vu} = \begin{bmatrix}K_{DL}^{LV} & K_{DV}^{LV}\\ K_{BL}^{LV} & K_{BV}^{LV}\end{bmatrix} \tag{30}$$

The latter matrix cannot be inverted since $K_{DL}^{LV} = -K_{BL}^{LV}$ and $K_{DV}^{LV} = -K_{BV}^{LV}$, as shown by Eqs. 5.

This is a trivial example, but it illustrates how one, through a simple systematic approach, can avoid implementing infeasible control systems.

Consistency Relationships in Distillation Control Systems

In the previous section was derived a general expression, Eq. 24, for transformations between models for different control structures for the same basic distillation system. If the different models do not follow the transformation, they are not consistent with each other.

The subject of this section is another type of consistency, i.e., consistency between the gains in the model for a single structure. Such consistency relationships can be derived from basic physical laws, as the mass and energy conservation laws. This has usually been overlooked in the literature [a rare exception being recent work by Skogestad and Morari (1987a)], which has resulted in inconsistent models. It also means that the modeling possibilities have not been fully utilized.

A general consistency relationship

Consider a process described by a steady-state model in the form of Eq. 18. Suppose that also some other relationships between the variables of the process are known. Denote these relationships by

$$\boldsymbol{c}(\boldsymbol{y}, \boldsymbol{v}, \boldsymbol{u}) = \boldsymbol{0} \tag{31}$$

or in linearized form

$$\boldsymbol{F}_{cy}\Delta\boldsymbol{y} + \boldsymbol{F}_{cv}\Delta\boldsymbol{v} + \boldsymbol{F}_{cu}\Delta\boldsymbol{u} = \boldsymbol{0} \tag{32}$$

Inserting Eq. 18 and taking into account that the relationships hold regardless of the value of $\Delta\boldsymbol{\mu}$, which is a vector of independent variables, give the consistency relationship

$$\boldsymbol{F}_{cy}\boldsymbol{K}_{yu} + \boldsymbol{F}_{cv}\boldsymbol{K}_{vu} + \boldsymbol{F}_{cu} = 0 \tag{33}$$

If $\boldsymbol{F}_{cy}$, $\boldsymbol{F}_{cv}$ and $\boldsymbol{F}_{cu}$ are known (which they are if Eq. 31 is obtained from first principles), Eq. 33 is a consistency relationship between $\boldsymbol{K}_{yu}$ and $\boldsymbol{K}_{vu}$.

Suppose that a model is known in the form of Eq. 19, where $\Delta\boldsymbol{\nu}$ and $\Delta\boldsymbol{\mu}$ are related to $\Delta\boldsymbol{v}$ and $\Delta\boldsymbol{u}$ by Eq. 21. One way of obtaining the consistency relationships for this model is by (an inverse) transformation into a model with $\Delta\boldsymbol{v}$ and $\Delta\boldsymbol{u}$ as variables, Eq. 18, and inserting the expressions for the process gains of this model into Eq. 33. This results in a rather complicated expression for the consistency relationships.

Another way is to express the known relationships, Eq. 32, directly in terms of the variables of the model in question. This can be done by inverting the variable transformation, Eq. 21, and eliminating $\Delta\boldsymbol{u}$ and $\Delta\boldsymbol{v}$ from Eq. 32 by the expression obtained. Taking the model, Eq. 19, into account, then gives the consistency relationship

$$\boldsymbol{F}_{cy}\boldsymbol{K}_{y\mu} + \boldsymbol{F}_{c\nu}\boldsymbol{K}_{\nu\mu} + \boldsymbol{F}_{c\mu} = 0 \tag{34}$$

where

$$[\boldsymbol{F}_{c\mu}\,\boldsymbol{F}_{c\nu}] = [\boldsymbol{F}_{cu}\,\boldsymbol{F}_{cv}]\begin{bmatrix} \boldsymbol{H}_{\mu u} & \boldsymbol{H}_{\mu v} \\ \boldsymbol{H}_{\nu u} & \boldsymbol{H}_{\nu v} \end{bmatrix}^{-1} \tag{35}$$

When the gains of the model are estimated, e.g., from experimental or simulated data, they should be estimated subject to the constraints imposed on them by the consistency relationship (Eq. 34). An appealing way to do this is first to estimate the gains in the usual way from a number of experiments without taking the consistency relationships into account. Then the gains, which usually are inconsistent due to experimental errors or nonlinear process behavior, can be reconciled subject to the consistency relationships, possibly by taking the accuracy of the initial estimates into account (Häggblom, 1987).

The inverse in Eq. 35 can, of course, be numerically evaluated, but usually it is quite easy to obtain an analytical solution by taking the sparsity of the matrix into account. In the examples that will follow, no inversion is even needed, because the variables of the known relationships, Eq. 32, can simply be regrouped according to the needs of the models.

In the previous section, y and x could be any type of output variables. In this section, y and x will specifically denote product compositions. This is when consistency relationships are most interesting. (If the output variables are not product compositions, there also exist consistency relationships, but they are rather trivial. They are obtained from the total material balance, Eq. 1a.)

(L, V) structure

The model for the (L, V) structure is:

$$\begin{bmatrix} \Delta y \\ \Delta x \end{bmatrix} = \begin{bmatrix} K_{yL}^{LV} & K_{yV}^{LV} \\ K_{xL}^{LV} & K_{xV}^{LV} \end{bmatrix}\begin{bmatrix} \Delta L \\ \Delta V \end{bmatrix} \tag{36a}$$

$$\begin{bmatrix} \Delta D \\ \Delta B \end{bmatrix} = \begin{bmatrix} K_{DL}^{LV} & K_{DV}^{LV} \\ K_{BL}^{LV} & K_{BV}^{LV} \end{bmatrix}\begin{bmatrix} \Delta L \\ \Delta V \end{bmatrix} \tag{36b}$$

For the separation process studied, Figure 1, the material balances valid at steady state are:

$$D + B = F \tag{37a}$$

$$Dy + Bx = Fz \tag{37b}$$

To be compatible with the model, Eq. 36, the material balances have to be linearized and deviation variables have to be introduced. In accomplishing this it should be noted that the model does not account for disturbances in F and z, which means that we are not, in this context, interested in how these affect the system. This means that F and z can be assumed to be constant. For the case where disturbances in F and z are included as inputs in the model, it is straightforward to derive consistency relationships accounting for this (Häggblom and Waller, 1988).

Linearization of Eqs. 37 around the nominal steady-state, with F and z constant, gives

$$\begin{bmatrix} 0 & 0 \\ \bar{D} & \bar{B} \end{bmatrix}\begin{bmatrix} \Delta y \\ \Delta x \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ \bar{y} & \bar{x} \end{bmatrix}\begin{bmatrix} \Delta D \\ \Delta B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{38}$$

Inserting the model for the (L, V) structure and taking into account that ΔL and ΔV are independent inputs, give the following consistency relationships for the (L, V) structure:

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ \bar{D} & \bar{B} & \bar{y} & \bar{x} \end{bmatrix} \begin{bmatrix} K_{yL}^{LV} & K_{yV}^{LV} \\ K_{xL}^{LV} & K_{xV}^{LV} \\ K_{DL}^{LV} & K_{DV}^{LV} \\ K_{BL}^{LV} & K_{BV}^{LV} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{39}$$

Equation 39 comprises four independent consistency relationships between the eight process gains. This means that four process gains can be expressed in terms of the other four process gains. However, not any four gains can be expressed in terms of the other four. The gains in Eq. 36a cannot be determined from the gains in Eq. 36b since the submatrix made up of columns 1 and 2 in the leftmost matrix in Eq. 39 is not invertible. On the other hand, the consistency relationships can be explicitly solved for the gains corresponding to any other set of two columns in the matrix. Solving with respect to the gains corresponding to columns 3 and 4 ($\bar{y} \neq \bar{x}$ if any separation is taking place) gives

$$\begin{bmatrix} K_{DL}^{LV} & K_{DV}^{LV} \\ K_{BL}^{LV} & K_{BV}^{LV} \end{bmatrix} = - \begin{bmatrix} 1 & 1 \\ \bar{y} & \bar{x} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ \bar{D} & \bar{B} \end{bmatrix} \begin{bmatrix} K_{yL}^{LV} & K_{yV}^{LV} \\ K_{xL}^{LV} & K_{xV}^{LV} \end{bmatrix}$$

$$= \frac{1}{\bar{y} - \bar{x}} \begin{bmatrix} -\bar{D} & -\bar{B} \\ \bar{D} & \bar{B} \end{bmatrix} \begin{bmatrix} K_{yL}^{LV} & K_{yV}^{LV} \\ K_{xL}^{LV} & K_{xV}^{LV} \end{bmatrix}$$

or

$$K_{DL}^{LV} = -K_{BL}^{LV} = -\frac{\bar{D}K_{yL}^{LV} + \bar{B}K_{xL}^{LV}}{\bar{y} - \bar{x}} \tag{40a}$$

$$K_{DV}^{LV} = -K_{BV}^{LV} = -\frac{\bar{D}K_{yV}^{LV} + \bar{B}K_{xV}^{LV}}{\bar{y} - \bar{x}} \tag{40b}$$

which are identical to Eqs. 5.

Equations 40 show that, if the material balances are taken into account, it is sufficient to describe the steady-state behavior of the (L, V) structure with the model expressed by Eq. 36a when the outputs are the product compositions.

(D, V) structure

The (D, V) structure is described by the static model

$$\begin{bmatrix} \Delta y \\ \Delta x \end{bmatrix} = \begin{bmatrix} K_{yD}^{DV} & K_{yV}^{DV} \\ K_{xD}^{DV} & K_{xV}^{DV} \end{bmatrix} \begin{bmatrix} \Delta D \\ \Delta V \end{bmatrix} \tag{41a}$$

$$\begin{bmatrix} \Delta L \\ \Delta B \end{bmatrix} = \begin{bmatrix} K_{LD}^{DV} & K_{LV}^{DV} \\ K_{BD}^{DV} & K_{BV}^{DV} \end{bmatrix} \begin{bmatrix} \Delta D \\ \Delta V \end{bmatrix} \tag{41b}$$

The linearized material balances with F and z constant, Eq. 38, can be rewritten in the form

$$\begin{bmatrix} 0 & 0 \\ \bar{D} & \bar{B} \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & \bar{x} \end{bmatrix} \begin{bmatrix} \Delta L \\ \Delta B \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \bar{y} & 0 \end{bmatrix} \begin{bmatrix} \Delta D \\ \Delta V \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{42}$$

Combining with Eq. 41 and considering that D and V are mutually independent inputs, give the following consistency relationships:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ \bar{D} & \bar{B} & 0 & \bar{x} \end{bmatrix} \begin{bmatrix} K_{yD}^{DV} & K_{yV}^{DV} \\ K_{xD}^{DV} & K_{xV}^{DV} \\ K_{LD}^{DV} & K_{LV}^{DV} \\ K_{BD}^{DV} & K_{BV}^{DV} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \bar{y} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{43}$$

In this case the submatrix made up of columns 3 and 4 in the leftmost matrix in Eq. 43 is not invertible. This means that the system is not fully determined by Eq. 41a. In the (D, V) structure the gains due to inventory control cannot, in contrast with the (L, V) structure, be determined from the gains for the product compositions.

For the (D, V) structure, the gains corresponding to columns 1 and 4 or 2 and 4 in the leftmost matrix in Eq. 43 can be explicitly solved from the consistency relationships. This also implies which gains should be given to define all eight gains in the model (Eqs. 41). In all cases the gains K_{LD}^{DV} and K_{LV}^{DV} have to be given in order to determine the system. However, to our knowledge they are never included in the models for the (D, V) structure given in the literature. Instead the assumptions

$$K_{LD}^{DV} = -1 \tag{44a}$$

$$K_{LV}^{DV} = 1 \tag{44b}$$

are made when the values of these gains are needed. These assumptions are equivalent to the assumptions expressed by Eqs. 6.

Solving Eq. 43 for the gains corresponding to, e.g., columns 2 and 4, gives the following explicit consistency relationships for the (D, V) structure:

$$K_{xD}^{DV} = -\frac{\bar{D}}{\bar{B}}\left(K_{yD}^{DV} + \frac{\bar{y} - \bar{x}}{\bar{D}}\right) \tag{45a}$$

$$K_{xV}^{DV} = -\frac{\bar{D}}{\bar{B}} K_{yV}^{DV} \tag{45b}$$

$$K_{DB}^{DV} = -1 \tag{45c}$$

$$K_{BV}^{DV} = 0 \tag{45d}$$

(L, B) structure

The other material balance structure, the (L, B) structure, is described by the static model

$$\begin{bmatrix} \Delta y \\ \Delta x \end{bmatrix} = \begin{bmatrix} K_{yL}^{LB} & K_{yB}^{LB} \\ K_{xL}^{LB} & K_{xB}^{LB} \end{bmatrix} \begin{bmatrix} \Delta L \\ \Delta B \end{bmatrix} \tag{46a}$$

$$\begin{bmatrix} \Delta D \\ \Delta V \end{bmatrix} = \begin{bmatrix} K_{DL}^{LB} & K_{DB}^{LB} \\ K_{VL}^{LB} & K_{VB}^{LB} \end{bmatrix} \begin{bmatrix} \Delta L \\ \Delta B \end{bmatrix} \tag{46b}$$

Combining the linearized material balances, Eq. 38, with Eqs. 46, and taking into account that L and B are mutually independent inputs, give

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ \bar{D} & \bar{B} & \bar{y} & 0 \end{bmatrix} \begin{bmatrix} K_{yL}^{LB} & K_{yB}^{LB} \\ K_{xL}^{LB} & K_{xB}^{LB} \\ K_{DL}^{LB} & K_{DB}^{LB} \\ K_{VL}^{LB} & K_{VB}^{LB} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & \bar{x} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{47}$$

or explicitly for the gains corresponding to columns 2 and 3 in the leftmost matrix:

$$K_{xL}^{LB} = -\frac{\bar{D}}{\bar{B}} K_{yL}^{LB} \tag{48a}$$

$$K_{xB}^{LB} = -\frac{\bar{D}}{\bar{B}} \left(K_{yB}^{LB} - \frac{\bar{y} - \bar{x}}{\bar{D}} \right) \tag{48b}$$

$$K_{DL}^{LB} = 0 \tag{48c}$$

$$K_{DB}^{LB} = -1 \tag{48d}$$

In the (L, B) structure the effect of the manipulators L and B on V, i.e., the gains K_{VL}^{LB} and K_{VB}^{LB}, must be given in order to determine the system. These gains cannot be calculated from the other gains in the system model (Eq. 46). Assumptions as $K_{VL}^{LB} = 1$ and $K_{VB}^{LB} = -1$ are equivalent to the generally false assumptions of Eqs. 6 and 44.

Process Gain Calculation and Reconciliation

If the process gains of one distillation control structure are known, the gains of any feasible control structure can be calculated by means of Eq. 24. This means that a large number of control structures can be investigated once the gains of one structure are known.

The following examples illustrate the transformation from the energy balance structure (L, V) into the material balance structure (D, V) as well as the accuracy of the often used "constant molar overflow" assumptions $\Delta D = -\Delta L$ for $V =$ constant and $\Delta D = \Delta V$ for $L =$ constant.

Columns used for illustration

McAvoy and Weischedel (1981) have compared energy balance and material balance control of three columns, labeled A, B and C. Column A is a benzene-toluene column with 13 trays, column B a benzene-toluene column with 17 trays, and column C a methanol-ethanol column with 27 trays. McAvoy and Weischedel modeled the columns using "standard" assumptions, but not the constant molar overflow assumption (McAvoy, 1983). By numerical simulations they obtained the following process gains for the (L, V) structure and the (D, V) structure at the nominal steady states given.

Column A ($\bar{z} = 0.50, \bar{y} = 0.93, \bar{x} = 0.07, \overline{L/D} = 1.30$):

$$\begin{bmatrix} \Delta y \\ \Delta x \end{bmatrix} = \begin{bmatrix} 0.580 & -0.450 \\ 0.350 & -0.480 \end{bmatrix} \begin{bmatrix} \Delta L \\ \Delta V \end{bmatrix}$$

$$\begin{bmatrix} \Delta y \\ \Delta x \end{bmatrix} = \begin{bmatrix} -0.710 & 0.140 \\ -0.420 & -0.130 \end{bmatrix} \begin{bmatrix} \Delta D \\ \Delta V \end{bmatrix}$$

Column B ($\bar{z} = 0.50, \bar{y} = 0.98, \bar{x} = 0.02, \overline{L/D} = 1.71$):

$$\begin{bmatrix} \Delta y \\ \Delta x \end{bmatrix} = \begin{bmatrix} 0.562 & -0.516 \\ 0.344 & -0.394 \end{bmatrix} \begin{bmatrix} \Delta L \\ \Delta V \end{bmatrix}$$

$$\begin{bmatrix} \Delta y \\ \Delta x \end{bmatrix} = \begin{bmatrix} -0.805 & 0.055 \\ -0.465 & -0.055 \end{bmatrix} \begin{bmatrix} \Delta D \\ \Delta V \end{bmatrix}$$

Column C ($\bar{z} = 0.50, \bar{y} = 0.99, \bar{x} = 0.01, \overline{L/D} = 4.51$):

$$\begin{bmatrix} \Delta y \\ \Delta x \end{bmatrix} = \begin{bmatrix} 0.471 & -0.495 \\ 0.749 & -0.832 \end{bmatrix} \begin{bmatrix} \Delta L \\ \Delta V \end{bmatrix}$$

$$\begin{bmatrix} \Delta y \\ \Delta x \end{bmatrix} = \begin{bmatrix} -0.685 & 0.0085 \\ -0.505 & -0.0094 \end{bmatrix} \begin{bmatrix} \Delta D \\ \Delta V \end{bmatrix}$$

McAvoy (1983) has later improved the accuracy of the gains in the material balance schemes. McAvoy also introduced D/F and V/F as manipulators instead of D and V, which gave the process gains a different scaling with different units. As the changes do not affect the main results of this study, we use the gains above since they are expressed in units more suitable for our analysis.

Numerical values for $\bar{D}$ and $\bar{B}$ are not given by McAvoy and Weischedel (1981), but can be estimated as follows. Since the separations are symmetrical, $\bar{D} = \bar{B}$. The consistency relationship (Eq. 45a) can then be used to estimate $\bar{D}$ (and $\bar{B}$). Equation 45a indicates the $\bar{D}$ and $\bar{B}$ would be in the range of 0.76 to 0.82 (expressed in units consistent with the units of the gains). In the sequel, the value $\bar{D} = \bar{B} = 0.75$ is used in accordance with a communication from McAvoy (1986), where he affirms that $\bar{D} + \bar{B} = 1.50$ was used in the study.

Constant molar overflow assumption

The process gains and the steady-state data given above can be used to check the often used assumption of constant molar overflow, Eqs. 6. Equations 40 give the following results for the three columns.

	Column		
	A	B	C
$K_{DL}^{LV} = -K_{BL}^{LV}$	−0.81	−0.71	−0.93
$K_{DV}^{LV} = -K_{BV}^{LV}$	0.81	0.71	1.02

Three other examples. An experimental methanol-water separation studied by Wood and Berry (1973) had $K_{DL}^{LV} = -0.25$. The same column studied at another steady state (Morris et al., 1982) had $K_{DL}^{LV} = -0.3$. An experimental ethanol-water distillation studied by Waller et al. (1988) had $K_{DL}^{LV} = -0.6$.

For more realistic distillation column models the assumption $\Delta D = -\Delta L$ (for $V =$ constant) and $\Delta D = \Delta V$ (for $L =$ constant) can thus be considerably erroneous. This also provides the answer to a question by Hashimoto (1981) in the discussion after presenting the paper by McAvoy and Weischedel.

Process gain calculation

The relations between the gains of the (L, V) structure and the (D, V) structure are given by Eq. 26a. Hence, this equation can be used to calculate the process gains of the (D, V) structure from the process gains of the (L, V) structure. The result for the three columns is as follows.

Column A:

$$\begin{bmatrix}\Delta y\\ \Delta x\end{bmatrix}=\begin{bmatrix}-0.715 & 0.130\\ -0.432 & -0.130\end{bmatrix}\begin{bmatrix}\Delta D\\ \Delta V\end{bmatrix}$$

Column B:

$$\begin{bmatrix}\Delta y\\ \Delta x\end{bmatrix}=\begin{bmatrix}-0.794 & 0.048\\ -0.486 & -0.048\end{bmatrix}\begin{bmatrix}\Delta D\\ \Delta V\end{bmatrix}$$

Column C:

$$\begin{bmatrix}\Delta y\\ \Delta x\end{bmatrix}=\begin{bmatrix}-0.504 & 0.017\\ -0.802 & -0.017\end{bmatrix}\begin{bmatrix}\Delta D\\ \Delta V\end{bmatrix}$$

The differences between the gains obtained by McAvoy and Weischedel through simulation and those calculated by means of the exact transformations are reasonable for columns A and B, but strikingly large for column C. In fact, if variable pairing for material balance control of column C were made on the basis of relative gain analysis, the calculated process gains would indicate another pairing than the simulated gains do (Häggblom and Waller, 1986). The explanation for the discrepancies is that the process gains obtained through simulation are not exact, which probably is due to nonlinear process behavior. This, of course, applies to the gains of both the (L, V) structure and the (D, V) structure.

The consistency of the gains in the (D, V) structure can also be checked by means of Eqs. 45.

Process gain reconciliation

To obtain consistent process gains for different control structures, one should determine the process gains subject to the transformation (Eq. 24) with the transformation matrices $\boldsymbol{H}$ appropriately defined. For the (L, V) and (D, V) structure this means that the process gains should be estimated subject to Eq. 26 and, if the primary output variables are product compositions, Eqs. 40.

One way of doing this is first to obtain estimates of the process gains by some standard method, then to reconcile the process gains by optimization subject to the known relationships. The criterion to be minimized can be chosen, e.g., as the sum of the squares of the relative adjustment of each gain.

Reconciliation of the process gains given by McAvoy and Weischedel (1981) for the (L, V) and (D, V) structure for columns A, B, and C gives, with the criterion mentioned, the following consistent process gains.

Column A:

$$\begin{bmatrix}\Delta y\\ \Delta x\end{bmatrix}=\begin{bmatrix}0.582 & -0.450\\ 0.344 & -0.480\end{bmatrix}\begin{bmatrix}\Delta L\\ \Delta V\end{bmatrix}$$

$$\begin{bmatrix}\Delta y\\ \Delta x\end{bmatrix}=\begin{bmatrix}-0.720 & 0.135\\ -0.426 & -0.135\end{bmatrix}\begin{bmatrix}\Delta D\\ \Delta V\end{bmatrix}$$

Column B:

$$\begin{bmatrix}\Delta y\\ \Delta x\end{bmatrix}=\begin{bmatrix}0.571 & -0.516\\ 0.338 & -0.394\end{bmatrix}\begin{bmatrix}\Delta L\\ \Delta V\end{bmatrix}$$

$$\begin{bmatrix}\Delta y\\ \Delta x\end{bmatrix}=\begin{bmatrix}-0.804 & 0.055\\ -0.476 & -0.055\end{bmatrix}\begin{bmatrix}\Delta D\\ \Delta V\end{bmatrix}$$

Column C:

$$\begin{bmatrix}\Delta y\\ \Delta x\end{bmatrix}=\begin{bmatrix}0.529 & -0.567\\ 0.612 & -0.675\end{bmatrix}\begin{bmatrix}\Delta L\\ \Delta V\end{bmatrix}$$

$$\begin{bmatrix}\Delta y\\ \Delta x\end{bmatrix}=\begin{bmatrix}-0.606 & 0.0089\\ -0.701 & -0.0089\end{bmatrix}\begin{bmatrix}\Delta D\\ \Delta V\end{bmatrix}$$

The reconciliation procedure is explained in more detail in Häggblom (1987), where it is applied to process gains obtained experimentally on a pilot-plant distillation column.

Process Dynamics

The transformation between distillation control structures has been derived for the process gains, i.e., for the steady-state part of the transfer functions describing the process dynamics.

The transformation is valid also for the dynamic case provided that the column inventory is perfectly controlled (Häggblom, 1986). In theory an assumption of perfect control in a dynamic sense is certainly questionable, but in practice it is often a good assumption for control loops that are much faster than the other control loops in the process. In distillation this is usually the case for the inventory control loops.

Example for illustration

Bequette and Edgar (1986) have modeled a 17-plate benzene-toluene separation using standard assumptions, including perfect level control and constant molar overflow. The assumption of perfect level control means that the transformation (Eq. 24) is exact also when the gain matrices $\boldsymbol{K}$ are replaced by transfer function matrices $\boldsymbol{G}$.

Bequette and Edgar modeled the column by a state space model from which they identified two transfer function models, one for the (L, V) structure and one for the (D, V) structure. The models are

$$\begin{bmatrix}\Delta x_3\\ \Delta x_{17}\end{bmatrix}=\begin{bmatrix}G_{yL}^{LV} & G_{yV}^{LV}\\ G_{xL}^{LV} & G_{xV}^{LV}\end{bmatrix}\begin{bmatrix}\Delta L\\ \Delta V\end{bmatrix} \tag{49}$$

with

$$\begin{bmatrix} G_{yL}^{LV} & G_{yV}^{LV} \\ G_{xL}^{LV} & G_{xV}^{LV} \end{bmatrix} = \begin{bmatrix} \dfrac{2.454e^{-s}}{20s+1} & \dfrac{-3.493e^{-s}}{22s+1} \\ \dfrac{3.0e^{-s}}{22s+1} & \dfrac{-5.519e^{-s}}{20s+1} \end{bmatrix} \tag{50}$$

and

$$\begin{bmatrix} \Delta x_3 \\ \Delta x_{17} \end{bmatrix} = \begin{bmatrix} G_{yD}^{DV} & G_{yV}^{DV} \\ G_{xD}^{DV} & G_{xV}^{DV} \end{bmatrix} \begin{bmatrix} \Delta D \\ \Delta V \end{bmatrix} \tag{51}$$

with

$$\begin{bmatrix} G_{yD}^{DV} & G_{yV}^{DV} \\ G_{xD}^{DV} & G_{xV}^{DV} \end{bmatrix} = \begin{bmatrix} \dfrac{-1.438e^{-s}}{20s+1} & \dfrac{0.3986e^{-s}}{5s+1} \\ \dfrac{-1.917e^{-s}}{20s+1} & \dfrac{-0.3329e^{-s}}{4s+1} \end{bmatrix} \tag{52}$$

The outputs are the mole fractions of the more volatile component on plates 3 and 17, which we denote by y and x, respectively.

Since the outputs are not product compositions, the previously derived consistency relationships between gains in a single model do not apply. However, the general transformation, Eq. 24, is valid. This means that the relation between the transfer functions in Eqs. 49 and 51 is given by the following equation, equivalent in form to Eq. 26a:

$$\begin{bmatrix} G_{yD}^{DV} & G_{yV}^{DV} \\ G_{xD}^{DV} & G_{xV}^{DV} \end{bmatrix} = \frac{1}{G_{DL}^{LV}} \begin{bmatrix} G_{yL}^{LV} & -(G_{yV}^{LV} G_{DV}^{LV} - G_{yV}^{LV} G_{DL}^{LV}) \\ G_{xL}^{LV} & -(G_{xL}^{LV} G_{DV}^{LV} - G_{xV}^{LV} G_{DL}^{LV}) \end{bmatrix} \tag{53}$$

After rearrangement this equation gives the following four relationships:

$$G_{DL}^{LV} = \frac{G_{yL}^{LV}}{G_{yD}^{DV}} \tag{54a}$$

$$G_{DL}^{LV} = \frac{G_{xL}^{LV}}{G_{xD}^{DV}} \tag{54b}$$

$$G_{DV}^{LV} = \frac{G_{yV}^{LV} - G_{yV}^{DV}}{G_{yD}^{DV}} \tag{55a}$$

$$G_{DV}^{LV} = \frac{G_{xV}^{LV} - G_{xV}^{DV}}{G_{xD}^{DV}} \tag{55b}$$

Inserting the numerical data of Eqs. 50 and 52 into Eqs. 54 gives

$$G_{DL}^{LV} = -1.706 \tag{56a}$$

$$G_{DL}^{LV} = -1.706\,\frac{20s+1}{22s+1} \tag{56b}$$

Inserting the data into Eqs. 55 gives

$$G_{DV}^{LV} = 2.706\,\frac{(20s+1)(6.74s+1)}{(22s+1)(4s+1)} \tag{57a}$$

$$G_{DV}^{LV} = 2.705\,\frac{(2.97s+1)}{(4s+1)} \tag{57b}$$

As seen from Eqs. 56 and 57, the models used by Bequette and Edgar (1986) are consistent in the static sense and as consistent as can be expected, when simple transfer function models are used, in the dynamic sense.

However, the process gains $K_{DL}^{LV} = -1.7$ and $K_{DV}^{LV} = 2.7$ are not the ones one might expect, i.e., $K_{DL}^{LV} = -1.0$ and $K_{DV}^{LV} = 1.0$, considering that the constant molar overflow assumption was used in the modeling. The explanation for this apparent inconsistency is that the transfer functions are given for scaled variables (Bequette and Edgar, 1986). Thus it is meaningless to speak of constant molar overflow with the implication that Eqs. 6 will hold unless the scaling is given. All cases are covered simply by specifying K_{DL}^{LV} and K_{DV}^{LV}.

Decoupling by Output Transformations

Transformation of output variables may be of interest for several reasons. One is that it is desirable to find (and control) outputs that eliminate or reduce the interaction between control loops. This subject is reviewed by Waller (1986) and studied through simulation by Waller and Finnerman (1987).

In this section output transformations as a means of eliminating interaction is treated. The starting point is a scheme discussed by McAvoy (1983).

Proposed output transformation of the (D, V) structure

To achieve one-way steady-state decoupling (i.e, triangular system matrix at zero frequency) McAvoy suggests a control scheme where the manipulators are D and V and where the controlled outputs, instead of y and x, are

$$\eta = \frac{z}{1-z}\cdot y + x \tag{58a}$$

$$\xi = \frac{y}{1-y}\cdot\frac{1-x}{x} \tag{58b}$$

(McAvoy denotes the controlled outputs by ξ and S, respectively, but we use the notation above in accordance with the general treatment below.)

McAvoy (1983) shows by means of an approximate analytical model that η is almost independent of V, which means that the system is practically one-way decoupled at steady-state when η is controlled by D. McAvoy (1983, p. 154) further notes that the variable transformations do not given perfect two-way decoupling, and that the analytical model he uses is too complicated for calculation of a general transformation by which two-way decoupling would be achieved.

The properties of the control scheme suggested can be analyzed by considering only the variable transformations and the existing consistency relationships.

Linearization of Eqs. 58 with z constant gives

$$\Delta\eta = \frac{\bar{z}}{1-\bar{z}}\Delta y + \Delta x \tag{59a}$$

$$\Delta\xi = \frac{\bar{\xi}}{\bar{y}(1-\bar{y})}\Delta y - \frac{\bar{\xi}}{\bar{x}(1-\bar{x})}\Delta x \tag{59b}$$

Combination with the model for the standard (D, V) structure,

$$\begin{bmatrix}\Delta y\\ \Delta x\end{bmatrix} = \begin{bmatrix}K_{yD}^{DV} & K_{yV}^{DV}\\ K_{xD}^{DV} & K_{xV}^{DV}\end{bmatrix}\begin{bmatrix}\Delta D\\ \Delta V\end{bmatrix} \tag{60}$$

gives the model

$$\begin{bmatrix}\Delta \eta\\ \Delta \xi\end{bmatrix} = \begin{bmatrix}K_{\eta D}^{DV} & K_{\eta V}^{DV}\\ K_{\xi D}^{DV} & K_{\xi V}^{DV}\end{bmatrix}\begin{bmatrix}\Delta D\\ \Delta V\end{bmatrix} \tag{61}$$

where

$$K_{\eta D}^{DV} = \frac{\bar{z}}{1-\bar{z}}K_{yD}^{DV} + K_{xD}^{DV} \tag{62a}$$

$$K_{\eta V}^{DV} = \frac{\bar{z}}{1-\bar{z}}K_{yV}^{DV} + K_{xV}^{DV} \tag{62b}$$

$$K_{\xi D}^{DV} = \frac{\bar{\xi}}{\bar{y}(1-\bar{y})}K_{yD}^{DV} - \frac{\bar{\xi}}{\bar{x}(1-\bar{x})}K_{xD}^{DV} \tag{62c}$$

$$K_{\xi V}^{DV} = \frac{\bar{\xi}}{\bar{y}(1-\bar{y})}K_{yV}^{DV} - \frac{\bar{\xi}}{\bar{x}(1-\bar{x})}K_{xV}^{DV} \tag{62d}$$

Taking into account the consistency relationships for the standard (D, V) structure, Eqs. 45, further gives

$$K_{\eta D}^{DV} = \left(\frac{\bar{z}}{1-\bar{z}} - \frac{\bar{D}}{\bar{B}}\right)K_{yD}^{DV} - \frac{\bar{y}-\bar{x}}{\bar{B}} \tag{63a}$$

$$K_{\eta V}^{DV} = \left(\frac{\bar{z}}{1-\bar{z}} - \frac{\bar{D}}{\bar{B}}\right)K_{yV}^{DV} \tag{63b}$$

$$K_{\xi D}^{DV} = \left(\frac{\bar{\xi}}{\bar{y}(1-\bar{y})} + \frac{\bar{\xi}}{\bar{x}(1-\bar{x})}\frac{\bar{D}}{\bar{B}}\right)K_{yD}^{DV} + \frac{\bar{\xi}}{\bar{x}(1-\bar{x})}\frac{\bar{y}-\bar{x}}{\bar{B}} \tag{63c}$$

$$K_{\xi V}^{DV} = \left(\frac{\bar{\xi}}{\bar{y}(1-\bar{y})} + \frac{\bar{\xi}}{\bar{x}(1-\bar{x})}\frac{\bar{D}}{\bar{B}}\right)K_{yV}^{DV} \tag{63d}$$

Equation 63b shows that $K_{\eta V}^{DV} = 0$ for separations with $\bar{z} = 0.5$ and $\bar{D} = \bar{B}$, or, more generally, for any separation where $\bar{D} = \bar{F}\bar{z}$ [and $\bar{B} = \bar{F}(1-\bar{z})$]. The latter condition is always at least approximately true for high-purity separations. In all these cases the system is thus one-way steady-state decoupled (because $K_{\eta D}^{DV}$ and $K_{\xi V}^{DV}$ are $\neq 0$ assuming $K_{yV}^{DV} \neq 0$). It is also apparent that in general the variable transformations suggested do not result in two-way decoupling, since $K_{\eta D}^{DV}$ will generally be $\neq 0$.

However, in general the variable transformations (Eq. 58) do not result in one-way decoupling either. A general output variable transformation that does result in perfect steady-state decoupling can be derived as follows.

General decoupling transformations of the (D, V) structure

Consider two transformations of the output variables y and x:

$$\eta = \eta(y, x) \tag{64a}$$

$$\xi = \xi(y, x) \tag{64b}$$

Linearization and introduction of deviation variables give the linear transformations

$$\Delta\eta = H_{\eta y}\Delta y + H_{\eta x}\Delta x \tag{65a}$$

$$\Delta\xi = H_{\xi y}\Delta y + H_{\xi x}\Delta x \tag{65b}$$

Combination with the model for the standard (D, V) structure gives the model expressed by Eq. 61, where now

$$K_{\eta D}^{DV} = H_{\eta y}K_{yD}^{DV} + H_{\eta x}K_{xD}^{DV} \tag{66a}$$

$$K_{\eta V}^{DV} = H_{\eta y}K_{yV}^{DV} + H_{\eta x}K_{xV}^{DV} \tag{66b}$$

$$K_{\xi D}^{DV} = H_{\xi y}K_{yD}^{DV} + H_{\xi x}K_{xD}^{DV} \tag{66c}$$

$$K_{\xi V}^{DV} = H_{\xi y}K_{yV}^{DV} + H_{\xi x}K_{xV}^{DV} \tag{66d}$$

Clearly any two variable transformations that satisfy the relations

$$\frac{H_{\eta y}}{H_{\eta x}} = -\frac{K_{xV}^{DV}}{K_{yV}^{DV}} \tag{67a}$$

$$\frac{H_{\xi y}}{H_{\xi x}} = -\frac{K_{xD}^{DV}}{K_{yD}^{DV}} \tag{67b}$$

will steady-state decouple η from V and ξ from D, respectively. Taking into account the consistency relationships for the standard (D, V) structure, Eqs. 45, then gives

$$\frac{H_{\eta y}}{H_{\eta x}} = \frac{\bar{D}}{\bar{B}} \tag{68a}$$

$$\frac{H_{\xi y}}{H_{\xi x}} = \frac{\bar{D}}{\bar{B}}\left(1 + \frac{\bar{y}-\bar{x}}{\bar{D}K_{yD}^{DV}}\right) \tag{68b}$$

Other, equivalent, expressions for the decoupling conditions are obtained by, e.g., the substitution

$$\frac{\bar{D}}{\bar{B}} = \frac{\bar{z}-\bar{x}}{\bar{y}-\bar{z}} \tag{69}$$

which follows from the material balances for the process, Eqs. 1.

This analysis shows that any variable transformation (Eq.

64a) such that the linearized form (Eq. 65a) satisfies Eq. 68a will steady-state decouple η from V. Likewise, any variable transformation (Eq. 64b) such that the linearized form (Eq. 65b) satisfies Eq. 68b will steady-state decouple ξ from D.

The two nonzero process gains in the model for the (D, V) structure with both outputs transformed so as to achieve two-way decoupling can be expressed, e.g., in the form

$$K^{DV}_{\eta D} = -H_{\eta y}\frac{\bar{y}-\bar{x}}{\bar{D}} \tag{70a}$$

$$K^{DV}_{\xi V} = -H_{\xi x}\frac{(\bar{y}-\bar{x})}{\bar{D}}\frac{K^{DV}_{xV}}{K^{DV}_{yD}} \tag{70b}$$

These expressions are obtained by combination of the decoupling conditions, Eqs. 68, and the consistency relationships, Eqs. 45, with the expressions for the gains in the transformed model, Eqs. 66.

The equations above do not give unique transformations for two-way steady-state decoupling, they only define necessary conditions which the transformations must satisfy. There is still freedom, e.g., to choose nonlinear transformations that counteract the nonlinear behavior of the process (i.e., transformations that linearize the process), or transformations that take dynamic effects into account (e.g., time delays).

However, if only static linear transformations are considered, then Eqs. 65a and 68a define a general output variable transformation that steady-state decouples η from V, regardless of the distillation process—no separation model is needed. There is still one degree of freedom, which can be used to give the output variable η or, equivalently, the process gain $K^{DV}_{\eta D}$ a suitable scaling by specifying, e.g., $H_{\eta y}$. The transformation is close to the one suggested by McAvoy (1983), Eq. 59a, especially for high-purity separations. This can be seen if Eq. 68a, with $H_{\eta x} = 1$, and Eq. 69 are inserted in Eq. 65a, thus giving

$$\Delta\eta = \frac{\bar{z}-\bar{x}}{\bar{y}-\bar{z}}\Delta y + \Delta x \tag{71}$$

Similarly, Eqs. 65b and 68b define a static linear output variable transformation that steady-state decouples ξ from D. Here, however, a model for the separation process is needed. The minimum information required is the process gain K^{DV}_{yD}, as shown by Eq. 68b. To obtain a general transformation, which could be used without explicit knowledge of process gains, one would have to know K^{DV}_{yD} as function of the process variables at steady state. Also in this case there is a degree of freedom, which can be used to give the output variable ξ, or the process gain $K^{DV}_{\xi V}$, a suitable scaling.

It is interesting to note that the condition for decoupling ξ from D can also be expressed in the form

$$\frac{H_{\xi y}}{H_{\xi x}} = \frac{\bar{D}}{\bar{B}}\left(1-\frac{1}{\lambda^{DV}_{yD}}\right) \tag{72}$$

where λ^{DV}_{yD} is the relative gain for the ($y - D$) pairing in the standard (D, V) structure. This expression is obtained through combination of Eq. 68b with Eq. 80, which is derived in the next section. Combining Eqs. 68a and 72 with Eqs. 65 and choosing $H_{\eta y}$ and $H_{\xi x}$ as scaling parameters, give the variable transformations

$$\Delta\eta = H_{\eta y}\Delta y + \frac{\bar{B}}{\bar{D}}H_{\eta y}\Delta x \tag{73a}$$

$$\Delta\xi = \frac{\bar{D}}{\bar{B}}\left(1-\frac{1}{\lambda^{DV}_{yD}}\right)H_{\xi x}\Delta y + H_{\xi x}\Delta x \tag{73b}$$

It follows from Eqs. 70 and 80 that the choice $H_{\eta y} = H_{\xi x} = \lambda^{DV}_{yD}$ will make the gains $K^{DV}_{\eta D}$ and $K^{DV}_{\xi V}$ in the transformed structure equal to the gains K^{DV}_{yD} and K^{DV}_{xV}, respectively, in the untransformed structure.

The relative gain λ^{DV}_{yD} for the reconciled models of column A, B and C is 0.628, 0.628 and 0.464, respectively (cf. Eq. 77). According to Eqs. 73, the columns can then be two-way steady-state decoupled by the following output variable transformations (note that $\bar{D} = \bar{B}$ for all three columns).

Column A:

$$\Delta\eta = 0.628\Delta y + 0.628\Delta x$$

$$\Delta\xi = -0.372\Delta y + 0.628\Delta x$$

Column B:

$$\Delta\eta = 0.628\Delta y + 0.628\Delta x$$

$$\Delta\xi = -0.372\Delta y + 0.628\Delta x$$

Column C:

$$\Delta\eta = 0.464\Delta y + 0.464\Delta x$$

$$\Delta\xi = -0.536\Delta y + 0.464\Delta x$$

These results should be viewed against several suggestions in the literature to control sums and differences so as to achieve decoupling (Waller and Finnerman, 1987).

General decoupling transformations of the (L,B) structure

The case where the composition manipulators are D and V was treated above. If the manipulators are L and B, the derivation of transformations for noninteraction is analogous to the derivation in the case with D and V as manipulators. The result is that η can be steady-state decoupled from B and ξ steady state decoupled from L by transformations according to Eqs. 65, with

$$\frac{H_{\eta y}}{H_{\eta x}} = \frac{\bar{D}}{\bar{B}}\left(1-\frac{\bar{y}-\bar{x}}{\bar{D}K^{LB}_{yB}}\right) \tag{74a}$$

$$\frac{H_{\xi y}}{H_{\xi x}} = \frac{\bar{D}}{\bar{B}} \tag{74b}$$

If $H_{\eta y}$ and $H_{\xi x}$ are chosen as scaling parameters, the nonzero gains in the model for the transformed (L, B) structure will be

$$K^{LB}_{\eta L} = H_{\eta y}\frac{(\bar{y}-\bar{x})K^{LB}_{yL}}{\bar{B}K^{LB}_{xB}} \tag{75a}$$

$$K^{LB}_{\xi B} = H_{\xi x}\frac{\bar{y} - \bar{x}}{\bar{B}} \tag{75b}$$

For the (L, B) structure, transformations expressed in terms of the relative gain λ^{LB}_{yL} are obtained by means of the expression for λ^{LB}_{yL}, Eq. 82, and the consistency relationships for the standard (L, B) structure, Eqs. 48.

Relative Gain Analysis

The relative gain concept, introduced by Bristol (1966) and developed, e.g., by Shinskey (1984) and McAvoy (1983), is a measure of interaction between control loops. Basically, relative gain analysis is used to pair inputs and outputs in a control structure, but based on the calculated interaction indices it is often also used for choosing the control structure (i.e., the inputs and outputs to be used) from a number of alternatives.

In its usual form, the relative gain analysis is a steady-state analysis and thus based on process gains only. This means that the transformations and consistency relationships derived above are readily applicable. If the process gains of a system are given for one control structure, such as the conventional (L, V) structure, the process gains for any other feasible control structure can be calculated. The relative gain for any distillation control structure can thus be calculated from known process gains for one structure.

In the (L, V) structure the relative gain for the $(y - L)$ pairing is given by

$$\lambda^{LV}_{yL} = \frac{1}{1 - \dfrac{K^{LV}_{xL}K^{LV}_{yV}}{K^{LV}_{yL}K^{LV}_{xV}}} \tag{76}$$

Analogously, the relative gains for the $(y - D)$ pairing in the (D, V) structure is given by

$$\lambda^{DV}_{yD} = \frac{1}{1 - \dfrac{K^{DV}_{xD}K^{DV}_{yV}}{K^{DV}_{yD}K^{DV}_{xV}}} \tag{77}$$

and the relative gain for the $(y - L)$ pairing in the (L, B) structure by

$$\lambda^{LB}_{yL} = \frac{1}{1 - \dfrac{K^{LB}_{xL}K^{LB}_{yB}}{K^{LB}_{yL}K^{LB}_{xB}}} \tag{78}$$

If the output variables y and x denote product compositions, the consistency relationships expressed by Eqs. 40, 45 and 48 apply. If one takes into account the relationships existing between the process gains of the (L, V) structure and the (D, V) structure, Eq. 26, and the consistency relationships for the (L, V) structure, Eqs. 40, λ^{DV}_{yD} can be expressed by the process gains of the (L, V) structure as

$$\lambda^{DV}_{yD} = \frac{1}{1 + \dfrac{\bar{B}K^{LV}_{xL}}{\bar{D}K^{LV}_{yL}}} \tag{79}$$

Another expression for λ^{DV}_{yD} is obtained if one inserts the consistency relationships for the (D,V) structure, Eqs. 45, in Eq. 77. The expression is

$$\lambda^{DV}_{yD} = -\frac{\bar{D}}{\bar{y} - \bar{x}}K^{DV}_{yD} \tag{80}$$

When the output variables are product compositions it is thus enough to know the process gain K^{DV}_{yD} and nominal steady-state data to be able to calculate the relative gains of the (D, V) structure.

Similarly, λ^{LB}_{yL} for the (L, B) structure can be expressed as

$$\lambda^{LB}_{yL} = \frac{1}{1 + \dfrac{\bar{D}K^{LV}_{yV}}{\bar{B}K^{LV}_{xV}}} \tag{81}$$

and

$$\lambda^{LB}_{yL} = \frac{\bar{B}}{\bar{y} - \bar{x}}K^{LB}_{xB} \tag{82}$$

The transformation for the (L, B) structure that corresponds to the transformation for the (D, V) structure (Eq. 26a) is

$$\begin{bmatrix} K^{LB}_{yL} & K^{LB}_{yB} \\ K^{LB}_{xL} & K^{LB}_{xB} \end{bmatrix} = \frac{1}{K^{LV}_{DV}}\begin{bmatrix} K^{LV}_{yL}K^{LV}_{DV} - K^{LV}_{yV}K^{LV}_{DL} & -K^{LV}_{yV} \\ K^{LV}_{xL}K^{LV}_{DV} - K^{LV}_{xV}K^{LV}_{DL} & -K^{LV}_{xV} \end{bmatrix} \tag{83}$$

From Eqs. 76, 77, 78, and the transformations (Eqs. 26a and 83), it follows that the relative gains λ^{LV}_{yL}, λ^{DV}_{yD} and λ^{LB}_{yL} are related by the expression

$$\left(1 - \frac{1}{\lambda^{LV}_{yL}}\right) = \left(1 - \frac{1}{\lambda^{DV}_{yD}}\right)\left(1 - \frac{1}{\lambda^{LB}_{yL}}\right) \tag{84}$$

Equation 84 has previously been derived by Jafarey et al. (1979) with the use of the constant molar overflow assumption $\Delta D = \Delta V - \Delta L$. As no limiting assumptions were made in the derivation above, the relationship is exact for any separation process, whether the assumption of constant molar overflow holds or not and regardless of, e.g., the type of outputs used.

Mijares et al. (1985) have proposed a method, based on the "Jacobian method of determinants," for evaluating the relative gains of nonlinear systems. The method was motivated by the need to reduce the number of simulations required to evaluate different control structures. The method is basically a numerical method which does not, e.g., take into account the consistency relationships that exist due to the material balances.

Although Mijares et al. (1985) have performed the simulations and subsequent calculations with great numerical accuracy, the omission of the consistency relationships has resulted in inconsistent estimates of the process gains. Mijares et al. do not report the obtained gains, but the consistency can be checked from the relative gains they report for different control structures.

For the separation studied, Mijares et al. have calculated the following relative gains for the (L, V), (D, V) and (L, B) structure:

$$\lambda^{LV}_{yL} = 11.45, \quad \lambda^{DV}_{yD} = 0.984, \quad \lambda^{LB}_{yL} = 0.021$$

These values do not satisfy Eq. 84. If the discrepancy is attributed to the value of λ_{yD}^{DV}, it is numerically quite small, but it corresponds to considerable discrepancies in the process gains. If consistency relationships had been taken into account in the estimation of process gains, this discrepancy could have been avoided.

Conclusions

Some fundamental properties of the continuous distillation process have been derived. Emphasis is put on the steady-state part of the transfer functions describing the process.

A general expression, Eq. 24, is derived for transformation of the gains of one (arbitrary) control structure into the gains of any feasible control structure. The transformation procedure is illustrated by two examples, where the material balance structure (D, V) and Ryskamp's structure $[D/(L + D), V]$ are derived from the conventional structure (L, V). It is also shown that transformations into control structures that are infeasible from a steady-state point of view are impossible.

Generally eight process gains are needed to describe the process for a binary separation into two product streams. These are the four gains between the two free manipulators and the two outputs to be controlled (e.g., composition or temperature) and the four gains between the same manipulators and the manipulators used for inventory control.

Due to constraints on the process, imposed, e.g., by material balances at steady-state operation, there are a number of consistency relationships between the eight gains. A general expression for the consistency relationships is derived. Explicit expressions for some common control structures, namely the conventional (L, V) structure and the two material balance structures (D, V) and (L, B), are also given.

It is shown that when process gains are estimated, existing consistency relationships should be explicitly taken into account, otherwise the estimated process gains will almost certainly be inconsistent due to nonlinear process behavior or measurement errors. An appealing way of doing this is first to estimate the process gains by standard methods, then to reconcile the obtained process gains subject to the consistency relationships.

If the process gains of one control structure are known, it is possible to calculate the process gains of any feasible control structure by means of analytical expressions for control structure transformations. It is thus necessary to determine process gains from simulated or experimental data for just one control structure: other control structures can be investigated through control structure transformations. However, if process gains for more than one control structure are estimated, e.g., in order to get more reliable estimates for a nonlinear or noisy process, they should be estimated subject to the analytical relationships that exist between the process gains of the control structures in question.

The expressions for the control structure transformations can also be used in the dynamic case provided that the secondary output variables, e.g., column inventory in a distillation process, are perfectly controlled in the control structures considered. If the manipulators for these variables are not changed by the transformation, perfect control is not even required. For distillation processes the assumption of perfect inventory control is often made, e.g. in simulation studies, although more realistic inventory control easily could be simulated. Use of the assumption in transformations of distillation control structure seems therefore justified.

The consistency relationships and control structure transformations are useful also for analytical purposes. By means of consistency relationships it is shown that the control loops in the (D, V) structure and the (L, B) structure can be one-way steady state decoupled by suitable transformations of the output variables. The transformations, valid for any separation in a distillation column, require no process model. It is also clarified what information is needed to achieve two-way steady-state decoupling by transformation of the output variables.

Some results relevant to relative gain analysis are derived. The relative gains of the (D, V) and (L, B) structure can, e.g., be calculated if just one of the process gains and nominal steady-state data are known when the primary outputs are product compositions. It can be also shown that the relative gains of different control structures are directly related by simple analytical expressions. For example, a simple relationship exists between the relative gains of the (L, V), (D, V) and (L, B) structures.

The control structure transformations are, of course, also useful for other purposes than those illustrated in this paper. One appealing application is the calculation of such manipulators that result in both complete steady-state rejection of disturbances, e.g., in feed composition and feed flow rate, and, at the same time, steady-state decoupling of the two primary control loops. This and other possibilities are investigated at present.

It could be added that the basic ideas presented in the paper are general and applicable to other processes than distillation as well.

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Notation

B = bottoms flow rate
$\boldsymbol{c}$ = vector of relationships known from first principles, especially material balances
D = distillate flow rate
F = feed flow rate
$\boldsymbol{F}_{cb}$ = matrix of partial derivatives of a vector of consistency relationships, $\boldsymbol{c}$, with respect to a vector of variables, $\boldsymbol{b}$
G_{ab}^{cd} = transfer function from an input variable b to an output variable a in control structure (c, d)
$H_{\beta b}$ = partial derivative of a variable β with respect to a variable b in a variable transformation
$\boldsymbol{H}_{\beta b}$ = matrix of partial derivatives of a vector β with respect to a vector $\boldsymbol{b}$ in a variable transformation
K_{ab}^{cd} = process gain between an input variable b and an output variable a in control structure (c, d)
$\boldsymbol{K}_{ab}$ = matrix of process gains (i.e., partial derivatives) between the input vector $\boldsymbol{b}$ and the output vector $\boldsymbol{a}$ in the control structure defined by $\boldsymbol{b}$
L = reflux flow to column
R = $D/(L + D)$
$\boldsymbol{u}$ = vector of manipulators, especially $\boldsymbol{u} = [L\ V]^T$
V = vapor flow from reboiler to column (can also denote steam or heat flow to reboiler)
$\boldsymbol{v}$ = vector of manipulators used for inventory control, especially $\boldsymbol{v} = [D\ B]^T$ in the (L, V) structure
x = bottoms product composition (e.g., mole fraction of more volatile component) or any other variable related to the bottoms product (e.g., composition or temperature on a plate in the column)

y = top product composition (e.g., mole fraction of more volatile component) or any other variable related to the top product (e.g., composition or temperature on a plate in the column)

$\mathbf{y}$ = vector of outputs, especially $\mathbf{y} = [y\ x]^T$

z = feed composition (e.g., mole fraction of more volatile component)

Greek letters

Δ = deviation from a nominal steady-state value

η = output variable defined by a variable transformation

λ_{ab}^{cd} = relative gain for pairing output variable a and input variable b in control structure (c, d)

μ = vector of free manipulators

ν = vector of manipulators used for inventory control

ξ = output variable defined by a variable transformation

Others

$\bar{\ }$ = nominal steady-state value

^{T} = transpose of a vector or a matrix

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